The Uniqueness of the Strongly Regular Graph on 77 Points

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ABSTRACT

We show the uniqueness of the strongly regular graph with parameters $v = 77, k = 16, \lambda = 0, \mu = 4$ embedding it in the Higman–Sims graph as a second subconstituent, and indicate the existence of a sporadic geometry.

INTRODUCTION

All terms not defined here are standard and can be found, e.g., in [3]. {Note that we—in accordance with standard practice but unlike [3]—use v, k, λ (μ) for the number of vertices, the valency, and the number of common neighbors of two adjacent (nonadjacent) vertices, respectively, of a strongly regular graph.}

There is a unique extension of the projective plane $PG(2, 4)$, namely, the Steiner system $S(3, 6, 22)$ [i.e., 3-(22, 6, 1)] (cf. [6]). It has parameters $v = 22$, $k = 6$, $\lambda_3 = 1$, $\lambda_2 = 5$, $\lambda_1 = 21$, $\lambda_0 = 77$, and any two blocks meet in either 0 or 2 points. Viewing the design as a quasisymmetric 2-design we see that the graph with the blocks as vertices and pairs of disjoint blocks as edges is strongly regular. In fact this graph has parameters $v = 77$, $k = 16$, $\lambda = 0$, $\mu = 4$. It can be embedded in a graph on 100 vertices as follows: Take as vertices a symbol ∞, the 22 points, and the 77 blocks of the design. Let ∞ be adjacent to all points of the design and let a point be adjacent to the blocks containing it and let blocks be adjacent if they are disjoint. This defines a strongly regular graph with parameters $v = 100$, $k = 22$, $\lambda = 0$, $\mu = 6$ called the Higman–Sims graph. Its automorphism group contains the simple Higman–Sims group as an index two subgroup (cf. [5]). We shall prove that any strongly regular graph with 77 vertices (and hence $k = 16$, $\lambda = 0$, $\mu = 4$—this

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is the only possibility allowed by the known restrictions on parameters of strongly regular graphs) can be embedded as the set of non-neighbors of a given vertex in the Higman–Sims graph and hence is (isomorphic to) the graph described above.

As an illustration of the method let us first give a short proof of the unicity of a strongly regular graph with the parameters of the Higman–Sims graph (cf. [2, 4]).

Proposition. A strongly regular graph G with parameters (100, 22, 0, 6) is isomorphic to the Higman–Sims graph.

Lemma. G does not contain a $K_{3,3}$.

Proof. Let there be x_i vertices outside a $K_{3,3}$ adjacent to i vertices inside.

Counting points we see that $\Sigma_i\, x_i = 100 - 6 = 94$.

Counting edges we see that $\Sigma_i\, ix_i = 6.19 = 114$.

Counting paths of length two we see that $\Sigma_i \binom{i}{2} x_i = 6.3 = 18$.

Consequently $0 \leq \frac{1}{2}\Sigma(i - 1)(i - 2)\, x_i = 18 - 114 + 94 = -2$, a contradiction. ∎

Proof of Proposition. Select a vertex and call it ∞. Call its neighbors points and its non-neighbors blocks. Identify a block with the set of ($\mu =$) 6 points it is incident with. The preceding lemma shows that no two blocks have three points in common. From the parameters of the graph one sees immediately that the blocks form a design with parameters $v = 22$, $k = 6$, $\lambda_2 = 5$, $\lambda_1 = 21$, $\lambda_0 = 77$. Also, on the average each triple of vertices is covered once, and no triple is covered twice, so each triple is covered exactly once, i.e., $\lambda_3 = 1$. This proves that the design is $S(3, 6, 22)$. Adjacent blocks have no common points (since $\lambda = 0$), but in $S(3, 6, 22)$ each block is disjoint from 16 other blocks, so the adjacent blocks are exactly the disjoint blocks. This proves that G is the Higman–Sims graph. ∎

THE 77-GRAPH

Theorem. Let G be a strongly regular graph with parameters $v = 77$, $k = 16$, $\lambda = 0$, $\mu = 4$. Then G is isomorphic to the complement of the block graph of $S(3, 6, 22)$.

Proof.

(i) G does not contain $K_{3,3}$ or $K_{4,4} - 3K_2$.

Proof. Suppose G contains a subgraph with w vertices, E edges, and N nonedges. Then as before we find

$$\sum x_i = 77 - w, \quad \sum ix_i = 16w - 2E, \quad \sum \binom{i}{2} x_i = 4N - \sum_{j=1}^{w} \binom{d_j}{2}$$

where $d_1, \ldots, d_w$ is the degree sequence of the subgraph. Again it follows that

$$77 - 17w + 2E + 4N - \sum \binom{d_j}{2} \geqslant 0.$$

But for $K_{3,3}$ we have $w = 6$, $E = 9$, $N = 6$, degrees 3^6, and

$$77 - 17.6 + 2.9 + 4.6 - 6\binom{3}{2} = -1.$$

Similarly for $K_{4,4} - 3K_2$ we have $w = 8$, $E = 13$, $N = 15$, degrees $4^2 3^6$, and

$$77 - 136 + 26 + 60 - 12 - 18 = -3. \quad \blacksquare$$

Fix a vertex of G and call it ∞; call its 16 neighbors points, and its 60 non-neighbors blocks. Identify a block with the set of four adjacent points. The first part of (i) says that two nonadjacent blocks intersect in at most two points. Since $\lambda = 0$, adjacent blocks are disjoint. The blocks form a 2-design 2-(16, 4, 3) with parameters $\lambda_2 = 3$, $\lambda_1 = 15$, $\lambda_0 = 60$. The derived design on a fixed point x is a 1-(15, 3, 3) design, and the second part of (i) says that this is the generalized quadrangle GQ(2, 2). (As follows: the points of GQ are the points distinct from x; the lines of GQ are the $B \setminus \{x\}$ where B is a block containing x. Two lines intersect in at most one point, and if y is a point, L a line, and $y \notin L$ then if there are two lines L_1, L_2 through y intersecting L in z_1, z_2, respectively, then we find a $K_{4,4} - 3K_2$ on the vertices ∞, L, L_1, L_2; x, y, z_2, z_1. Consequently the twelve lines not containing y intersecting the three lines containing y are all distinct, and we do have a generalized quadrangle.)

Now GQ(2, 2) is unique and self-dual. One convenient representation for it is the following: let the points be the pairs from a fixed 6-set, and the lines the partitions of this 6-set into three pairs, with obvious incidence. From this representation one sees that there are six ovals (sets of five points, no two collinear), namely the six sets of pairs containing a given symbol. Any point is in two ovals and any two nonadjacent points determine a unique oval. By duality there are six spreads (parallel classes, sets of

five lines partitioning the point set); any line is in two sprea
two disjoint lines determine a unique spread.

The idea of the proof is to view the 2-(16, 4, 3) as a frag
3-(22, 6, 1) design and to find the missing points and bloc
moment we can define the missing points. Fix a point x_0. The
quadrangle determined by x_0 contains 6 spreads. Let these
points. Call a line of the quadrangle incident with (or adj
spread if it is in it. This takes care of 15 of the blocks. W
to find out how the remaining 45 blocks are related to the n

Let us first see what such a block looks like in the generalized
GQ.

(ii) There is a 1-1 correspondence between blocks B not c
and flags (y, M) in the GQ such that M is the unique line
nonadjacent to B and B is the symmetric difference of the l
L_2 passing through y and distinct from M. (B and M are p
77-graph and adjacency is adjacency in the graph.)

Proof. A block B not containing x_0 is a 4-subset of GQ.
the 2-(16, 4, 3) design we find that 12 blocks intersect B in
32 blocks intersect B in a single point so that 15 blocks are d
B. On the other hand, B is adjacent to 12 blocks, so there a
disjoint and nonadjacent to B. If 2 of these blocks had a point of
x, then in the corresponding GQ we have at least 2 lines d
the 4-set B, and nonadjacent to B. Also, x and B have μ =
neighbors; i.e., in this GQ there are exactly 4 lines adjac
hence disjoint from) B. This leaves at most $15 - 2 - 4$ =
intersect B, so that there are at least 3 lines L_i intersecting B
The three pairs $L_i \cap B$ cannot be pairwise disjoint; suppose
$B \neq \emptyset$, $L_1 \cap B = \{a, b\}$, $L_2 \cap B = \{a, c\}$. Then the secon
applied to the graph induced by $\infty, B, L_1, L_2; a, x, c, b$ yields a co

Consequently the 3 blocks disjoint and nonadjacent to B forr
of the remaining 12 points, and exactly one is a line in the GQ
by x_0. Reexamining the above argument, we now see that
lines intersect B, 8 in a single point, and 2, L_1 and L_2, say
The two pairs $B \cap L_1$ and $B \cap L_2$ are disjoint. The lines L_1 an
be disjoint, otherwise one of the 3 lines intersecting both L_1 a
contain 2 points of B. Hence L_1 and L_2 intersect in a point y
the third line through y. (See Fig. 1.)

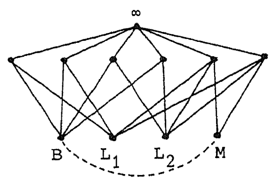

FIGURE 1.

All we have to show is that $B \nsim M$. Consider the subgraph induced by ∞, the 4 points of B, x_0, y, B, L_1, L_2, and M. If $B \sim M$ then we find [using the notation of the proof of (i)]

$$w = 11,\ E = 21,\ N = 34,\ \text{degrees } 6^1 5^1 4^4 3^5,$$

so that

$$\sum x_i = 66, \quad \sum i x_i = 134, \sum \binom{i}{2} x_i$$
$$= 136 - 15 - 10 - 24 - 15 = 72;$$

hence

$$\sum \frac{1}{2}(i-2)(i-3)\, x_i = 3x_0 + x_1 + x_4 + 3x_5 + \cdots = 2.$$

Consequently $x_1 \leqslant 2$. But ∞ has 16 neighbors, 10 outside this subgraph, and only 2 are adjacent to M, so $x_1 \geqslant 8$. This contradiction proves that $B \nsim M$. (And now there is a unique solution $x_0 = 0$, $x_1 = 8$, $x_2 = 46$, $x_3 = 12$.) [That the correspondence is 1-1 follows since there are 45 flags (y, M) in GQ.] ∎

[Note that we know the 2-(16, 4, 3) design at this point: the blocks passing through x_0 are the lines of the GQ and the blocks not passing through x_0 are the symmetric differences of two intersecting lines of the GQ. Or in other words (as remarked by A. Neumaier): the blocks are the 4-cliques in the unique regular two-graph on 16 points.

Next note (as observed by a referee) that adjacency between lines of the GQ and other blocks in determined now, but since any point can take the role of x_0 all adjacencies are determined so that there is at most one possibility for our 77-graph. Thus, our main theorem is proved already, and the rest of this note in fact constitutes a uniqueness proof for $S(3, 6, 22)$ relative to the uniqueness of GQ(2, 2).]

Having found this unique line M corresponding to B, we make B adjacent to the same two new points M is adjacent to.

(iii) Considering the blocks as 6-sets (with 4 old and 2 new points each), we have $B \cap B' \in \{0, 2\}$ and $B \cap B' = \emptyset$ iff $B \sim B'$ for all pairs B, B' of distinct blocks.

Proof. This is clear if both B and B' are lines from GQ.

Suppose $x_0 \notin B$, $x_0 \in B'$. Let B correspond to (y, M). If $B' = M$ then the statement is clear. If $B' = L_i$ $(i = 1, 2)$ also. If B' intersects B in a single point then B' and M are in a unique spread so that B' and B share a unique new point.

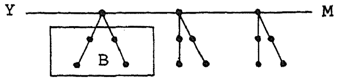

FIGURE 2.

Finally, if B' is disjoint from B and different from M then B' and B have no new points in common (since B' is one of the 4 lines intersecting M in a point other than y) and $B' \sim B$. (See Fig. 2.) Now let $x_0 \notin B$, $x_0 \notin B'$. Let B correspond to (y, M) and B' to (z, N). If B and B' have two new points in common (but $B \neq B'$) then $M = N$, $y \neq z$, and $B \cap B' = \emptyset$, and we have to show that $B \not\sim B'$. But if $M = \{x_0, y, z, u\}$ and L_5, L_6 are the lines through u different from M then L_5 and L_6 are common neighbors of B and B' so that $B \not\sim B'$. If B and B' have one new point in common then we have to show that they also have one old point in common. But "one new point in common" is equivalent to "M and N in a common parallel class," i.e., to M and N disjoint lines. Now y has a unique neighbor on N, say $N \cap L_1 = \{x, v\}$. If $v = z$ then $B \cap B' = L_1 \setminus \{y, z\}$, a singleton. If $v \neq z$ then z has a neighbor $w \neq y$ on L_2 and $B \cap B' = \{w\}$.

Finally, if B and B' do not have a new point in common, then M and N intersect in a point p. If $y = z = p$ and L is the third line through p then $B \cap B' = L \setminus \{p\}$. If $y \neq p$, $z \neq p$ then $|B \cap B'| = 2$ by the properties of a GQ. If $y \neq p$, $z = p$ or $y = p$, $z \neq p$ then $|B \cap B'| = 0$. In this case in fact $B \sim B'$: B has 16 neighbors; 4 are points, 4 are lines, and the remaining 8 must be found in this case. But there are exactly 8 possibilities for B' here. ∎

All that is missing now are some new blocks that should cover the triples which are not covered yet. But these are uniquely determined: all triples containing both old and new points have been covered already (this follows by counting). Add a new block Ω consisting of the 6 new points. A 6-set through a point x containing only uncovered triples is an oval in the GQ corresponding to x, and there are exactly 6 of them. This shows that there are $16.6/6 = 16$ 6-sets containing only uncovered triples. Take these also as new blocks (they will be neighbors of Ω). This completes the construction of the $S(3, 6, 22)$ design. Letting disjoint blocks be adjacent we find the Higman–Sims graph, and our original graph G consists of all non-neighbors of the vertex Ω. Since the Higman–Sims graph has a transitive group of automorphisms (this follows from the proof of the proposition) this shows that G is isomorphic to the graph on the non-neighbors of ∞, i.e., the complement of the block graph of $S(3, 6, 22)$. ∎

Remark. A. Neumaier observed that this proof shows that the 77-graph is an object with the Buekenhout–Tits diagram (Fig. 3) (cf. [1]). In fact, let the objects of type 1 be the 77 vertices of the graph, those of type

FIGURE 3.

2 the 2310 nonedges (pairs of nonadjacent vertices), those of type 3 the 2310 4-sets $\mu(p, q) := \{z | p \sim z \sim q\}$ (for nonadjacent p, q), and those of type 4 the 77 16-sets $\Gamma(p) := \{z | p \sim z\}$. Let incidence be inclusion.

[The starting point of the above proof is just the observation that the residue of a flag of type {1, 4} is a generalized quadrangle GQ(2, 2). All other checks are trivial. That the geometry satisfies the intersection axiom follows from the fact that the graph does not contain $K_{3,3}$.]

The automorphism group of the graph (Aut(M_{22})) acts transitively on maximal flags. The mapping interchanging p and $\Gamma(p)$, and $\{p, q\}$ and $\mu(p, q)$, is a polarity of the geometry.

This is the only geometry I know of with this diagram.

References

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